A Processor Core Model for Quantum Computing

Man-Hong Yung,[1,*] Simon C. Benjamin,[2,3] and Sougato Bose[4]

[1] *Physics Department, University of Illinois at Urbana-Champaign, Urbana IL 61801-3080, USA*
[2] *Department of Materials, Oxford University, Oxford OX1 3PH, United Kingdom*
[3] *Centre for Quantum Computation, Department of Physics, Oxford University OX1 3PU, United Kingdom*
[4] *Department of Physics and Astronomy, University College London, Gower Street, London WC1E 6BT*

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We describe an architecture based on a processing 'core' where multiple qubits interact perpetually, and a separate 'store' where qubits exist in isolation. Computation consists of single qubit operations, swaps between the store and the core, and *free* evolution of the core. This enables computation using physical systems where the entangling interactions are 'always on'. Alternatively, for switchable systems our model constitutes a prescription for optimizing many-qubit gates. We discuss implementations of the quantum Fourier transform, Hamiltonian simulation, and quantum error correction.

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Typically, schemes for solid state quantum computing involve an array of qubits with some form of direct physical interaction coupling nearby elements [1]. In order to implement a specific algorithm, these schemes require the experimentalist to dynamically control the magnitude of each qubit-qubit interaction - effectively to be able to switch it 'on' and 'off'. A common idea for achieving this is to somehow dynamically manipulate the wavefunction overlap between a pair of neighboring qubits, while other nearby qubits are decoupled. This appears feasible, but *highly* challenging. Moreover, even if a switching mechanism can be implemented, frequent switching is likely to increase the rate of dechoerence. A deeper objection is that, by having the majority of a system's interactions 'off' at a given moment, we are failing to maximally exploit its computational potential.

Recently ideas have emerged [2, 3] for computation in systems where the interaction remains always on. However, these proposals find ways to *effectively* pacify an interaction, and therefore one can make the same objection that they are not exploiting the full entangling power of the device. One class of system that does make full use of a set of permanent interactions is the mirror-inversion chain [4, 5, 6, 7, 8, 9]. A chain of spins, with suitably engineered coupling strengths, has the property that a qubit placed on one end will later manifest at the other - even though at intervening times it is distributed over the chain. When more than one qubit is placed on the chain, each will manifest at the complimentary site - but typically the qubits will have aquired an entangling phase. It has been observed [7] that this phase could in principle be employed to create certain classes of entangled state, graph states, which are the resource for one-way computation.

In this letter, we demonstrate the potential of such engineered spin chains to directly implement arbitrary controlled multi-qubit gates. The chain then acts as the computation core of our computer (see Figure 1) - we need only supplement its free evolution with swap operations and single qubit manipulations. Note that this model is profoundly distinct from schemes involving a single qubit *bus*, e.g. the original ion trap schemes, since there the common mode represents only one qubit of information. We show that a controlled multi-qubit gate can be constructed with exactly four free evolutions of the spin chain, independent of the number of spins involved. The controlling qubit can be any member of the spin chain, and the conditional unitary operations applied to the target qubits can be of any type. Such a gate can significantly reduce the number of elementary operations for quantum algorithms involving many non-local two-qubit operations.

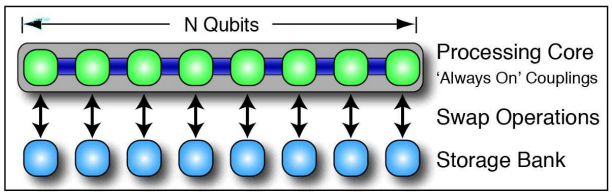

FIG. 1: Architecture of the processor core model. The core is an engineered spin-1/2 chain with always on interactions. The storage bank consists of isolated sites where qubits can be swapped to and from the corresponding sites in the processing core. Controlled multi-target gates are constructed by the free evolution of the spin chain.

We start with a finite chain of N spin-$\frac{1}{2}$ particles confied within local potentials and interaction with their nearest neighbors. The Hamiltonian considered is:

$$H = \frac{1}{2}\sum_{j=1}^{N-1}\omega_j\left(\sigma_j^x\sigma_{j+1}^x+\sigma_j^y\sigma_{j+1}^y\right)+\frac{1}{2}\sum_{j=1}^{N}\lambda_j\left(\sigma_j^z+1\right), \quad (1)$$

where the coupling constants $\{\omega_j, \lambda_j\}$ are real and in general distinct. We will adopt the convention that $|0_j\rangle$ ($|1_j\rangle$) refers to be the spin-down state $|\downarrow\rangle$ ($|\uparrow\rangle$) at the site j.

Next, we require that the core should possess the so-called "mirror symmetry" [5], which implies the coupling constants satisfy the relations $\omega_j = \omega_{N-j}$ and $\lambda_j = \lambda_{\bar{j}}$ where $\bar{j} \equiv N-j+1$ denotes the mirror-conjugate site of j. Let $|s\rangle \equiv |s_1 s_2 \cdots s_N\rangle$, $s_j = \{0,1\}$, be a particular spin configuration. Mirror inversion is said to occur when the state $|s\rangle$ is driven by the evolution operator $U(\tau) = e^{-iH\tau}$, for some fixed time period τ, to the inverted state $|\bar{s}\rangle \equiv |s_N \cdots s_2 s_1\rangle$ (up to a phase factor). Note that the term *mirror inversion* refers to the inversion of classical states $|s\rangle$ in which all the sites have definite spin values. For a quantum state, being a superposition of the basis states in general, some internal phases will be acquired. However, it is exactly these entangling phases which we will exploit for gate construction.

It is shown [8] that the sufficient and necessary condition for mirror inversion in mirror symmetrical chains is determined by the eigenvalue spectrum of H_S ($\hbar = 1$), the single excitation subspace of H. Then,

$$e^{-iE_k\tau} = (-1)^k e^{-i\phi_N} \quad , \tag{2}$$

where E_k, $k = 0, 1, 2, \cdots, N-1$, is the $(k+1)$-th eigenvalue of H_S and ϕ_N is some global phase independent of k. Due to the mirror symmetry, the coupling constants $\{\omega_j, \lambda_j\}$ can be determined by the eigenvalue spectrum. It is therefore an inverse eigenvalue problem [10]. Recently, many spectra [4, 5, 9] satisfying the condition in Eq.(2) have been proposed. However, to keep our model general, we will continue our discussion without reference to any specific type of spectrum.

To construct multi-qubit gates, we need to know the matrix elements of the evolution operator $U(\tau)$ in the $|s\rangle$ basis. Let $\mathsf{U} \equiv U(\tau)$. By mapping our picture of localized spins to that of spinless fermions [5, 8], one can show that

$$\mathsf{U}\,|s\rangle = e^{-ni\phi_N} (-1)^{(n-m)/2} |\bar{s}\rangle \quad , \tag{3}$$

where n is the number of spin-up states in $|s\rangle$, and m=0 (1) if n is even (odd). The factor $(-1)^{(n-m)/2}$ could be understood intuitively as follows: if n is even (i.e. m=0), then the operation of mirror inversion (reordering the state) is equivalent to swapping $n/2$ pairs of fermions. Similarly for odd n, except the factor should be the same as that of $n-1$ fermions. The phase factor $e^{-ni\phi_N}$ is a consequence of Eq.(2). In fact, the phase ϕ_N can be set to zero if an appropriate spectrum of H_S is chosen. In this case, it has been demonstrated [7] that the operator $U(\tau)$ alone can generate a fully connected graph state. However, in constructing multi-qubit gates, spurious correlations among qubits in the graph state have to be eliminated. This can be achieved with the help of an ancilla qubit within the storage array, initialized to be $|0\rangle_a$. Let S_x represent the swap operation between the spin at the site $\bar{x}$ (compliment of x) and the ancilla. We apply S_x to the state in Eq.(3) and allow the engineered chain to evolve once more, i.e. applying $\mathsf{Z}^x \equiv \mathsf{U}\mathsf{S}_x\mathsf{U}$ [11] to the initial state $|0\rangle_a \otimes |s\rangle$. Then from (3) the final state is

$$e^{-(2n-s_x)i\phi_N} (-1)^{s_x(n-1)} |s_x\rangle_a \otimes |s_1 s_2 \cdots 0_x \cdots s_N\rangle \ , \tag{4}$$

which is the same for n being either odd or even. Here we have only assumed a swap operation performed between the site $\bar{x}$ and the ancilla. Therefore, the qubit staying at the ancilla spin cannot be transferred back to the spin chain at this stage. However, as we shall see (cf. Eq.(7)), a more general multi-qubit gate can be constructed based on Z^x and all of the qubits can reside in their original locations at the end of the operation.

The next step is to interpret the result (4) in terms of the quantum circuit model. The phase factor $e^{-(2n-s_x)i\phi_N}$ can be regarded as a result of N local phase gates $\mathsf{R}_j(-2\phi_N)$, where $\mathsf{R}_j(\varphi) \equiv |0_j\rangle\langle 0_j| + e^{i\varphi}|1_j\rangle\langle 1_j|$, acting on all qubits, and one extra phase gate, $\mathsf{R}_j(\phi_N)$ acting on the spin at site x along. On the other hand, the factor $(-1)^{s_x(n-1)}$ can be considered as due to the application of controlled-σ_z to all qubits, except the spin at site x which is encoded with the controlling qubit. Suppose we now apply local operations to get rid all of the controlled phase gates R_j (or simply choose an eigenvalue spectrum such that $\phi_N = 0$), effectively we have constructed a controlled multi-target gate, which requires two free evolutions of the engineered Hamiltonian for any N. Note that for this multi-qubit gate generated by Z^x, the σ_z gate has to be applied to *all* qubits, controlled by a single qubit at site $\bar{x}$. However, the σ_z gate can be converted into controlled-V_j [12], where

$$\mathsf{V}_j = \begin{pmatrix} \sin\theta_j & e^{i\varphi_j}\cos\theta_j \\ e^{-i\varphi_j}\cos\theta_j & -\sin\theta_j \end{pmatrix} \quad , \tag{5}$$

through local operations A_j provided that the relations $\mathsf{A}_j\,\sigma_z\mathsf{A}_j^\dagger = \mathsf{V}_j$ and $\mathsf{A}_j\mathsf{A}_j^\dagger = \mathsf{I}_j$, where I_j is the identity operator, are satisfied. For example, the case $\theta_j = \varphi_j$=0 corresponds to a σ_x gate. Operationally, we denote the construction of this controlled multi-target gate by $\mathsf{V}^x \equiv \mathsf{A}\mathsf{Z}^x\mathsf{A}^\dagger$, where $\mathsf{A} \equiv \prod_{j=1}^N \mathsf{A}_j$ (and similarly for $\mathsf{A}^\dagger$). An immediate application of this gate is that, if we initialize the controlling qubit to be $|0\rangle + |1\rangle$ and the rest $|000...0\rangle$, it can efficiently generate a cat state $|000...0\rangle + |111...1\rangle$, which is interesting for various applications including single qubit measurement and encoding error correcting codes, such as the Shor's code.

The controlled operations V_j are not yet completely general: for example, the phase gate $\mathsf{R}_j(\varphi)$ and the identity operator I_j are excluded. We can construct a more general controlled multi-target gate, which applies arbitrary unitary operations W_j on the qubits. To proceed, consider applying U to the state $|s_1 s_2 \cdots 0_x \cdots s_N\rangle$, which is assumed to contain n spin-up states. The phase factors generated are exactly the same as that in Eq.(3).

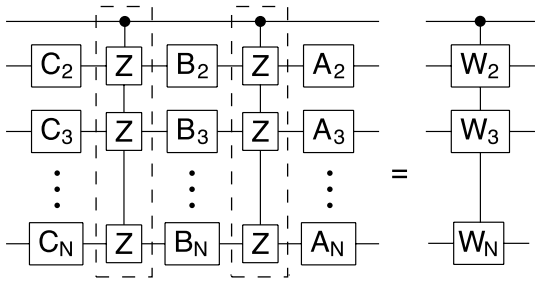

FIG. 2: The circuit diagram of a controlled multi-target gate $\mathsf{W}^1 = \mathsf{AZ}^1\mathsf{BZ}^1\mathsf{C}$ constructed by the free evolution of an engineered spin chain 'core'. The parts inside the dashed boxes are Z^1 which involves two free evolutions of the spin chain and one ancilla in the 'store'. The local operators A_j, B_j and C_j satisfy $\mathsf{A}_j\mathsf{B}_j\mathsf{C}_j = \mathsf{I}_j$ and $\mathsf{A}_j\mathsf{ZB}_j\mathsf{ZC}_j = \mathsf{W}_j$, where the unitary operations W_j, up to a phase, are defined in (6).

Now we can apply US_x to the resulting state; S_x returns $|s_x\rangle_a$ from the ancilla to the core. The final state is $e^{-(2n+s_x)i\phi_N}\,(-1)^{s_x n}\,|s\rangle$, which can also be considered as an controlled multi-target gate analogous to the one generated by $\mathsf{Z}^x\,|s\rangle$. For simplicity, we again assume $\phi_N = 0$. To construct a more general unitary matrix

$$\mathsf{W}_j = \begin{pmatrix} e^{i(\alpha_j-\beta_j-\delta_j)}\cos\gamma_j & -e^{i(\alpha_j-\beta_j+\delta_j)}\sin\gamma_j \\ e^{i(\alpha_j+\beta_j-\delta_j)}\sin\gamma_j & e^{i(\alpha_j+\beta_j+\delta_j)}\cos\gamma_j \end{pmatrix}, \quad (6)$$

including the identity operator, one can always choose a set of local operations A_j, B_j and C_j such that the relations $\mathsf{A}_j\mathsf{B}_j\mathsf{C}_j = \mathsf{I}_j$ and $e^{i\alpha_j}\mathsf{A}_j\mathsf{ZB}_j\mathsf{ZC}_j = \mathsf{W}_j$ [12] are satisfied. Our result is then that the controlled-W_j gate W^x, with controlling qubit at site x, can be constructed by the following sequence of operations: $\mathsf{W}^x = \mathsf{AZ}^x\mathsf{BZ}^x\mathsf{C}$, where A, B and C are the tensor products of local operators A_j, B_j and C_j respectively. Equivalently,

$$\mathsf{W}^x = |0_x\rangle\langle 0_x| \otimes \mathsf{I} + |1_x\rangle\langle 1_x| \otimes \prod_{j\neq x}\mathsf{W}_j \quad . \quad (7)$$

The circuit diagram for an example W^1 is shown in Figure 2. For any N, the cost of generating W^x includes four free evolutions of the engineered chain, two swap operations with the same ancilla and local operations.

We now describe the application of this model to the key problems of quantum Fourier transform, Hamiltonian simulation and quantum error correction. When we wish to make a statement about the efficiency of our *processor core* model, we will compare it with a notional *fully-switched* system having a Hamiltonian similar to (1), except that the interactions ω_i can be independently switched on and off. We assume the fully-switched system couples only pairs of qubits simultaneously, although it may do so in parallel - i.e. $\omega_i\omega_{i+1} = 0$ at all times. A real fully-switched system could, presumably, activate several adjacent interactions: the results we describe here can be seen as a prescription for doing precisely that, in order to increase efficiency. The primary gain in efficiency will of course be a reduction in the number of switching events (and consequently, a potential reduction in the decoherence rate) - but remarkably there can also be an absolute speed-up by a fixed factor, as we presently discuss.

One of the immediate applications of the controlled multi-target gate W^x is the operation of quantum Fourier transform (QFT), which is a key ingredient in many quantum algorithms such as the Shor's algorithm. In the above notations, the standard QFT circuit can be constructed by applying the multi-qubit gates and the Hadamard gates alternatively,

$$\mathsf{QFT} = \mathsf{H}_N\mathsf{W}^{N-1}\mathsf{H}_{N-1}\cdots\mathsf{W}^2\mathsf{H}_2\mathsf{W}^1\mathsf{H}_1 \quad , \quad (8)$$

where $\mathsf{W}_j = \mathsf{R}_j\left(\pi/2^{j-x}\right)$ for $j > x$ and $\mathsf{W}_j = \mathsf{I}_j$ otherwise. Here each joint operation W_j costs exactly four free evolutions, including two swaps. The QFT circuit depth is therefore $O(N)$, as is the total number of switching events. For the switched model, the circuit depth is also $O(N)$, while the absolute number of switching events is $O(N^2)$.

The second application is the simulation of the evolution of an "artificial" Hamiltonian H_A formally representing a joint interaction between r spin-1/2 particles,

$$H_A = \sigma_1^z \otimes \sigma_2^z \otimes \sigma_3^z \cdots \otimes \sigma_r^z \quad , \quad (9)$$

which is locally equivalent to the class of the Hamiltonian of the form $\sigma_1^{w_1}\otimes\sigma_2^{w_2}\otimes\sigma_3^{w_3}\cdots\otimes\sigma_r^{w_r}$, where $\sigma_j^{w_j} = \sigma_j^x$, σ_j^y or σ_j^z. Although it is unlikely to find a group of spin-1/2 particles interacting naturally under the Hamiltonian H_A, some higher dimensional systems can be mapped by those two-level systems. Moreover, the form of H_A can be considered as basic building blocks for simulating more complex Hamiltonians through local operations and the approximation: $e^{i(A+B)\Delta t} = e^{iA\Delta t}e^{iB\Delta t} + O\left(\Delta t^2\right)$ for short time-interval Δt.

Consider an engineered core of $N+1$ spins, with $\phi_N = 0$, initialized as $|0s\rangle \equiv |0s_1s_2...s_N\rangle$. The quantum circuit for simulating the evolution operator $U_A\left(\Delta t\right) = e^{-iH_A\Delta t}$ for (9) can be constructed by the following sequence of operations [12]:

$$U_A\left(\Delta t\right) = \mathsf{H}_0\mathsf{W}^0\mathsf{H}_0\mathsf{T}_0\left(\Delta t\right)\mathsf{H}_0\mathsf{W}^0\mathsf{H}_0 \quad , \quad (10)$$

where $\mathsf{T}_0\left(\Delta t\right) = \exp\left(-i\sigma_0^z\Delta t\right)$ and $W_j = \sigma_j^z$ (or $W_j = I_j$ if the qubit at site j is not involved). The basic idea of this construction is to store the parity (i.e. $m = \{0,1\}$) of the sites $j = 1,2,3,\ldots,N$ to site 0. The phase generated by the local operation T_0, depending on the parity, is exactly the one required for H_A. From (7), it is apparent that the series of non-local operations W^0 can be achieved by four free evolutions of the engineered chain of $N \geq r$ spins, with the aid of an ancilla and local operations. This scheme offers a flexibility of generating

interactions involving various number of spins using the same spin chain. The costs of generating each type of interactions are fixed (eight free evolutions). Alternatively, if one just needs to generate interactions with *fixed* number of qubits, i.e. $r = N$, the cost can be reduced to two free evolutions and no ancilla is needed. The sequence of operations in this case is

$$U_A(\Delta t) = \mathsf{H}_0\mathsf{U}\mathsf{H}_{\bar{0}}\mathsf{T}_{\bar{0}}(\Delta t)\,\mathsf{H}_{\bar{0}}\mathsf{U}\mathsf{H}_0 \quad . \tag{11}$$

The crucial observation for obtaining (11) is that $\mathsf{H}_{\bar{0}}\mathsf{U}\mathsf{H}_0\,|0s\rangle = (-1)^{(n-m)/2}\,|\bar{s}m\rangle$. Thus, the desired phase can be obtained by applying the local operator $\mathsf{T}_{\bar{0}}$ at site $\bar{0}$.

The processor core model will also be advantageous in running quantum algorithms in a fault tolerant fashion with concatenated code-words. For example, in the Steane code, six gates of the class W^x are required for error syndrome measurement (Fig.10.16 of Ref.[12]). In our approach, each level of concatenation just multiplies the number of targets in each Z^x by 7 [13] but does not increase the number of applications of such gates. However, the number of elementary switching operations required by a fully-switched system to realize W^x increases 7 fold with *each* level of concatenation.

The discussions above highlight potential gains in terms of simplicity: the circuit depth, or total number of switching operations. It is also interesting to ask, can the total time required be reduced by applying the processor core model? We can quickly conclude that any speed-up must be bounded, since the fundamental operation U (eqn. 3) can be simulated on a fully-switched array in time $O(N)$ [14], while U also takes time $O(N)$ to evolve on our processor-core (given a fixed maximum interaction strength). Interestingly, there can be speed-ups within this constraint, i.e. by bounded factors. To make a definite statement we specialize to a core with a linear spectrum, i.e. $\Delta_k = E_k - E_{k+1}$ being constant (e.g. [4, 5, 9]) since this is the time-optimal choice for a given spectral *range* [15]. Let us the compare the time required for a *simple state transfer*, i.e. $|100...0\rangle \rightarrow |0...001\rangle$, from one end of the chain to the other. If we now assume for simplicity that that the maximum interaction strength $\omega_{\max} \simeq N/4$ (see Ref. [4]) scales as N, then the time required for the processor-core evolution is simply π for all N. On the fully-switched system, the time required for each swap is $\frac{\pi}{2\omega_j}$, thus the total time required is $T(N) = \sum_{j=1}^{N-1} \frac{\pi}{2\omega_j}$. (Recall that our fully-switched system can couple only pairs of qubits simultaneously, thus each state transfer must complete before the next is initiated). One can easily show that $T(N) \geq (N-1)\,\pi/2\omega_{\max} \simeq 2\pi\,(N-1)/N$. Thus this *always-on* processing core can be superior by a factor of 2 for large N.

Finally, we remark that the periods of free processor core evolution can be relatively robust versus timing errors $\tau \rightarrow \tau + \delta t$ in subsequent swaps to the store. Consider the most general case, we start with the state $|\bar{\psi}\rangle = \sum_j \alpha_j\,|\bar{s}_j\rangle$, where $\sum_j |\alpha_j|^2 = 1$. If the evolution time is taken perfectly, we expect the final state to be $|\psi\rangle = \sum_j \alpha_j e^{i\phi_j}\,|s_j\rangle$, where ϕ_j represents the overall phase in Eq. (3) for the spin configuration $|s_j\rangle$. If not, we have $\langle\psi|\,U(\tau+\delta t)\,|\bar{\psi}\rangle = \sum_{j,k} \alpha_j^* \alpha_k e^{i(\phi_k - \phi_j)}\,\langle s_j|\,U(\delta t)\,|s_k\rangle \equiv 1 + iA\delta t + B\delta t^2 + O\left(\delta t^3\right)$. Here both A and B are real. The error, defined as $\epsilon \equiv 1 - \left|\langle\psi|\,U(\tau+\delta t)\,|\bar{\psi}\rangle\right|^2$, is therefore just second order in δt.

In conclusion, we have demonstrated how to construct controlled multi-target gates through the natural evolution of a processor core where interactions are *always on*. This model allows computation with physical systems where the entangling interactions are not switchable. Alternatively, in switchable systems our protocol can play an important role in simplifying multi-qubit operations. We demonstrated this by showing that the *fully-switched* model is fundamentally more complex for certain important algorithmic tasks. For various operations spanning many qubits, including long range state transfer, the temporal requirement for the processor core model is less-than-or-equal-to that of the fully-switched model. Therefore, in terms of the reduction of dynamical control while maintaining the same *speed*, many schemes presently described in terms of two-qubit gates can be enhanced by incorporating the processor core concept.

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* email: myung2@uiuc.edu

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operator by $Z_1 Z_2 .. Z_7$.

[14] One can simulate U on a 1D full-switched system using parallel swap operations performed over N time steps. One simply alternates between swaps on all pairs (s_{2i}, s_{2i+1}), and then all pairs (s_{2i-1}, s_{2i}). Each qubit will, at some point, be adjacent to every other, thus by interleaving nearest-neighbor phase gates with the swap operations, we implement the correct phase, still within $O(N)$ time steps. The total number of gates is $O(N^2)$.

[15] According to Eq. (2) the phase difference $\Delta_k \tau$ must be at least π for $k = 0, 1, 2, ..., N-2$. To compare different eigenvalue spectra, we require the range to be normalized $E_{\max} - E_{\min} = 1$. The linear spectrum requires the minimal evolution time to be $\tau = (N-1)\pi$. For any eigenvalue spectrum which is non-linear, there must exist at least one $\Delta_k < 1/(N-1)$, and hence $\tau > (N-1)\pi$.